Central Limit Theorem for Local Linear Statistics in Classical Compact Groups and Related Combinatorial Identities

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Abstract

We discuss CLT for the global and local linear statistics of random matrices from classical compact groups. The main part of our proofs are certain combinatorial identities much in the spirit of works by Kac and Spohn.

1   Introduction

Let M be a unitary matrix chosen at random with respect to the Haar measure on the unitary group $U(n)$. We denote the eigenvalues of M by $\{\exp(i \cdot \theta_j)\}_{j=1}^n$, where $-\pi \leq \theta_1, \theta_2, \ldots, \theta_n < \pi$. The joint distribution of the eigenvalues (called the Weyl measure) is absolutely continuous with respect to the Lebegue measure $\prod_{j=1}^n d\theta_j$ on the n-dimensional tori and its density

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is given by

$$P_{U(n)}(\theta_1, \dots, \theta_n) = \frac{1}{(2\pi)^n \cdot n!} \cdot \prod_{1 \leq j < k \leq n} |\exp(i \cdot \theta_j) - \exp(i \cdot \theta_k)|^2 \qquad (1.1)$$

(see [We]). Throughout the paper we will be interested in the global and local linear statistics

$$S_n(f) = \sum_{j=1}^{n} f(\theta_j), \qquad (1.2)$$

$$S_n(g(L_n \cdot)) = \sum_{j=1}^{n} g(L_n \cdot \theta_j), \qquad (1.3)$$

$$L_n \to \infty, \quad \frac{L_n}{n} \to 0.$$

The optimal conditions on f, g for our purposes are

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2 \cdot |k| < \infty, \qquad (1.4)$$

$$\int_{-\infty}^{\infty} |\hat{g}(t)|^2 \cdot |t| dt < \infty, \qquad (1.5)$$

where

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \cdot e^{ikx},$$

$$g(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \hat{g}(t) \cdot e^{itx} dt$$

However in order to simplify the exposition we will always assume that f has a continuous derivative on a unit circle ($f \in C^1(S^1)$) and g is a Schwartz function ($g \in f(J(\mathbb{R}^1))$).

Let us denote by E_n the mathematical expectation with respect to Haar measure. We start with the formulation of the result which is essentially due to C. Andréief ([A], for a modern day reference see [TW]and also [Dy].

Proposition

$$E_n \exp(tS_n(f)) - \det(Id + (e^{tf} - 1)K_n) = \det(Id + (e^{tf} - 1)Q_n), \quad (1.6)$$

where $(e^{tf} - 1)$ is a multiplicaiton operator and $K_n, Q_n : L^2(S^1) \to L^2(S^1)$ are the integral operators with the kernels

$$K_n(x, y) = \frac{1}{2\pi} \frac{\sin\left(\frac{n}{2}(x - y)\right)}{\sin\left(\frac{x-y}{2}\right)}, \quad (1.7)$$

$$Q_n(x, y) = \sum_{j=0}^{n-1} \frac{1}{\sqrt{2\pi}} e^{ijx} \frac{1}{\sqrt{2\pi}} e^{-ijy} \quad (1.8)$$

Remark 1. K_n, Q_n are unitary equivalent to each other and are the operators of a finite rank. In particular, Q_n is just a projection operator on the first n harmonic functions of the unit circle.

One of the ingredients of the proof of the proposition is the following chain of the equalities

$$
\begin{aligned}
p_{U(n)}(\theta_1, \dots, \theta_n) &= \frac{1}{n!} \cdot \det(e^{i \cdot (j-1) \cdot \theta_k})_{1 \le j,k \le n} \cdot \det(e^{-i \cdot (j-1) \cdot \theta_k})_{1 \le j,k \le n} \\
&= \frac{1}{n!} \det\left(Q_n(\theta_j, \theta_k) \right)_{1 \le j,k \le n} \\
&= \frac{1}{n!} \det\left(K_n(\theta_j, \theta_k) \right)_{1 \le j,k \le n}
\end{aligned}
\quad (1.9)
$$

Remark 1 allows us to rewrite the Fredholm determinants in (1.6) as the Toeplitz determinant with the symbol $\exp\big(t \cdot f(\cdot)\big)$:

$$
\begin{aligned}
E_n \exp\left(t \sum_{j=1}^{n} f(\theta_j) \right) &= D_{n-1}\left(\exp(t \cdot f) \right) \\
&= \det\left(\frac{1}{2\pi} \int_0^{2\pi} \exp\big(tf(x)\big) \cdot \exp\big(i(j-k)x\big) dx \right)_{1 \le j,k \le n}
\end{aligned}
\quad (1.10)
$$

The asymptotics of (1.10) for large n is given by the Strong Szego Limit Theorem:

$$D_{n-1}\bigg(\exp(t \cdot f)\bigg) =$$

$$\exp\bigg(tn\hat{f}(0) + \frac{1}{2}t^2 \sum_{-\infty}^{+\infty} |k||\hat{f}(k)|^2 + \bar{0}(1)\bigg) \tag{1.11}$$

(see [Sz] and [K], [H], [De], [F-H], [G-I], [Wid1], [Wid2], [McC-W], [Ba-W], [Jo1], [Bo], [Bo-S], [Me], [So2], [Wie], [D] for further developments.)

In probabilistic terms (1.11) claims that $ES_n(f) = \frac{n}{2\pi} \cdot \int_{-\pi}^{\pi} f(\theta)d\theta + \bar{0}(1)$ (actually the remainder term is zero), and the centralized random variable $\sum_{j=1}^{n} f(\theta_j) = E_n \sum_{j=1}^{n} f(\theta_j)$ converges in distribution to the normal law $N(0, \sum_{-\infty}^{\infty} |k||\hat{f}(k)|^2)$.

Our first goal is to establish a similar result for the local linear statistics.

Theorem 1. *Let $g \in J(\mathbb{R}^1)$, $L_n \to +\infty$, $\frac{L_n}{n} \to 0$. Then $E_n \sum_{j=1}^{n} g(L_n \cdot \theta_j) = \frac{n}{2\pi \cdot L_n} \cdot \int_{-\infty}^{\infty} g(x)dx$, and the centralized random variable $\sum_{j=1}^{n}(g(L_n \cdot \theta_j) - E \sum_{j=1}^{n} g(L_n\theta_j)$ converges in distribution to the normal law $N(0, \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} |\hat{g}(t)|^2 |t|dt)$.*

We give a combinatorial proof which holds both in the local and global cases. In some sense our approach is close to the heuristic arguments in [I-D]. We start with

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Lemma 1. *Let $C_{\ell,n}(f)$ be the ℓ-th cumulant of $S_n(f)$. Then*

$$
\left| C_{\ell,n}(f) - \sum_{k_1+\ldots+k_\ell=0} \hat{f}(k_1) \cdot \ldots \cdot \hat{f}(k_\ell) \cdot \sum_{m=1}^{\ell} \frac{(-1)^{m-1}}{m} \cdot \right.
$$

$$
\sum_{\substack{\ell_1+\ldots+\ell_m=\ell, \\ \ell_1 \geq 1,\ldots,\ell_m \geq 1}} \frac{\ell!}{\ell_1! \cdot \ldots \cdot \ell_m!} \cdot \left(n - \max(0, \sum_{i=1}^{\ell_1} k_i, \sum_{i=1}^{\ell_1+\ell_2} k_i, \ldots, \right.
$$

$$
\sum_{i=1}^{\ell_1+\ldots+\ell_{m-1}} k_i) - \max(0, \sum_{i=1}^{\ell_1}(-k_i), \sum_{i=1}^{\ell_1+\ell_2}(-k_i), \ldots,
$$

$$
\left. \left. \sum_{i-1}^{\ell_1+\ldots+\ell_{m-1}}(-k_i)) \right) \right| \leq \mathrm{const}_\ell \cdot \sum_{\substack{k_1+\ldots+k_\ell=0 \\ |k_1|+\ldots+|k_\ell|>n}} |k_1||\hat{f}(k_1)| \cdot \ldots \cdot |\hat{f}(k_\ell)|
$$

$$
\tag{1.12}
$$

Remark 2 One can see that for sufficiently smooth f the r.h.s. of (1.12) goes to zero as $n \to \infty$.

Remark 3 An analogous result to lemma 1 was established in [Spo] for the determinantal random point field with the sine kernel (see also Remark 4 below).

The proof of Lemma 1 will be given in §2. At this state we observe that it implies

Lemma 2 *The limit of $C_{\ell,n}(f), \ell > 1$ exists as $n \to \infty$ and is equal to $\sum_{k_1+\ldots+k_\ell=0} \hat{f}(k_1) \cdot \ldots \cdot \hat{f}(k_\ell) \cdot (G(k_1,\ldots,k_\ell) + G(-k_1,\ldots,-k_\ell))$, where G is the piece-wise linear continuous function defined by*

$$
G(k_1,\ldots,k_\ell) := \sum_{\sigma \in S_\ell} \sum_{m=1}^{\ell} \frac{(-1)^m}{m} \cdot \sum_{\substack{\ell_1+\ldots+\ell_m=\ell, \\ \ell_1 \geq 1,\ldots,\ell_m \geq 1}} \frac{1}{\ell_1! \cdot \ldots \cdot \ell_m!} \cdot
$$

$$
\max\left(0, \sum_{i=1}^{\ell_1} k_{\sigma(i)}, \sum_{i=1}^{\ell_1+\ell_2} k_{\sigma(i)}, \ldots, \sum_{i=1}^{\ell_1+\ldots+\ell_{m-1}} k_{\sigma(i)}\right).
$$

$$
\tag{1.13}
$$

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Proof of Lemma 2 After opening the brackets in (1.12) we observe that the coefficient in front of n is equal to

$$\sum_{m=1}^{\ell} \sum_{\substack{\ell_1 + \ldots + \ell_m = \ell, \\ \ell_1 \geq 1, i=1,\ldots,m}} \frac{(-1)^{m-1}}{m} \frac{\ell!}{\ell_1! \ldots \ell_m!} = \begin{cases} 1, & \ell = 1 \\ 0, & \ell > 1 \end{cases} \tag{1.14}$$

Indeed, the generating function of these coefficients is equal to

$$\log\left(1 + (e^z - 1)\right) = z.$$

$\square$

Now CLT for $\sum_{j=1}^{n} f(\theta_j)$ follows from

Main Combinatorial Lemma

Let $k_1, \ldots, k_\ell$ be arbitrary real numbers such that their sum equals zero. Let $G(k_1, \ldots, k_\ell)$ be defined as in (1.13). Then

$$G(k_1, \ldots, k_\ell) = \begin{cases} |k_1| = |k_2| & \text{if } \ell = 2 \\ 0 & \text{if } \ell > 2 \end{cases}.$$

We will prove the lemma in §3.

Remark 4 A similar combinatorial lemma was stated by Spohn in [Spo]. He studied a time-dependent motion of a system of infinite number of particles governed by the equations

$$d\lambda_j(t) = \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j} dt + db_j(t),$$

where $\{b_j(t)\}_{j=-\infty}^{+\infty}$-independent standard brownian motions, and the initial distribution of particles is given by determinantal random point field with the sine kernel $\frac{\sin \pi(x-y)}{\pi(x-y)}$. However, no correct proof of the combinatorial result was given there. For completeness we give a proof of Spohn's lemma independently from the proof of our Main Combinatorial Lemma in §3.

Assuming the combinatorial part is done we can quickly finish the proof of Theorem 1. The formula for the mathematical expectation is trivial. Rewriting (1.12) for the higher cumulants of $\sum_{j=1}^{n} g(L_n \cdot \theta_j)$ we see that the limit of the ℓ-th cumulant is given by

$$(2\pi)^{-\frac{\ell}{2}} \cdot \int \hat{g}(t_1) \cdot \ldots \cdot \hat{g}(t_\ell) \cdot \left(G(t_1, \ldots, t_\ell) + G(-t_1, \ldots, -t_\ell) \right) dt_1 \ldots dt_\ell$$

where the integral is over the hyperplane $t_1 + \ldots + t_\ell = 0$.

Theorem 1 is proven. □

Remark 5 Our method also gives an elementary combinatorial proof of Szegö theorem ((1.11)) for $f \in C^1(S^1)$ and sufficiently small complex t. It is different from the one suggested by Kac in [K] where the Taylor expansion of $D_n(1-tg)$ as a function of t was calculated and then a so- called Kac-Spitzer combinatorial lemma was employed to confirm (1.11).

Remark 6 Results similar to Theorem 1 have been established for other random matrix models in [Spo], [Jo3], [KKP], [Ba], [B-F], [SSo1], [SSo2], [BM-K].

The rest of the paper is organized as follows. We prove Lemma 1 in §2 and Main Combinatorial Lemma in §3. The result analogous to Theorem 1 for orthogonal and symplectic groups is established in §4.

2   Proof of Lemma 1

We start with calculating the moments of $S_n(f)$. Le us remember that k-point correlation function of the eigenvalues of random unitary matrix is given by

$$
\rho_{n,k}(\theta_1, \ldots, \theta_k) = \frac{n!}{(n-k)!} \int_{T^{n-k}} p_{U(n)}(\theta_1, \ldots, \theta_n) d\theta_{k+1} \ldots d\theta_n
$$
$$
= \det\left(K_n(\theta_i, \theta_j) \right)_{1 \le i,j \le k} = \det\left(Q_n(\theta_i, \theta_j) \right)_{1 \le i,j \le k}
$$

(2.1)

The N-th moment of $S_n(f)$ is equal to

$$
E_n \left(\sum_{i_1=1}^{n} f(\theta_{i_1}) \cdot \ldots \cdot \sum_{i_N=1}^{n} f(\theta_{i_N}) \right),
$$

where the indices $i_1, \ldots, i_N$ range independently from 1 to n, and in particular can coincide. Let $\mathcal{M} = \{M_1, \ldots, M_r\}$ be a partition of the set $\{1, 2, \ldots, N\}$ into subsets determined by coinciding indices among $i_1, \ldots, i_N$:

$$M_1 = \{j_1^{(1)}, \ldots, j_{s_1}^{(1)}\}, \ldots, M_r = \{j_1^{(r)}, \ldots, j_{s_r}^{(r)}\}, \sqcup_{i=1}^r M_i = \{1, 2, \ldots, N\},$$
$s_i = |M_i|, i = 1, \ldots r.$ Then

$$E_n \left(S_n(f) \right)^N = \sum_{\substack{\text{over all} \\ \text{partitions } \mathcal{M}}} E_n \sum_{\ell_1 \neq \ell_2 \neq \ldots \neq \ell_r} f^{s_1}(\theta_{\ell_1}) \cdot \ldots \cdot f^{s_r}(\theta_{\ell_r}) \qquad (2.2)$$

Let us consider a typical term in (2.2) corresponding to a partition $\mathcal{M}$.

$$E_n \sum_{\ell_1 \neq \ldots \neq \ell_r} f^{s_1}(\theta_{\ell_1}) \cdot \ldots \cdot f^{s_r}(\theta_{\ell_r}) =$$
$$\int_{T^r} f^{s_1}(x_1) \cdot \ldots \cdot f^{s_r}(x_r) \cdot \rho_{n,r}(x_1, \ldots, x_r) dx_1 \ldots dx_r \qquad (2.3)$$

By definition of the determinant and (2.1)

$$\rho_{n,r}(x_1, \ldots, x_r) = \sum_{\sigma \in S_r} (-1)^\sigma \prod_{i=1}^r Q_n(x_i, x_{\sigma(i)}).$$

Writing the permutation $\sigma \in S_r$ as a product of cyclic permutations we have

$$\rho_{n,r}(x_1, \ldots, x_r) = \sum_{\substack{\text{over partitions} \\ \mathcal{K} \text{ of } \{1, \ldots r\}}} \left(\prod_{\alpha=1}^q \left((-1)^{p_\alpha - 1} \cdot \right. \right.$$
$$\left. \left. \sum_{\substack{\text{over all cyclic} \\ \text{permutations of } K_\alpha}} \prod_{j=1}^{p_\alpha} Q_n\left(x_{t_j^{(\alpha)}}, x_{\sigma(t_j^{(\alpha)})}\right) \right) \right) \qquad (2.4)$$

where $\{1, \ldots, r\} = \sqcup_1^q K_\alpha, K_\alpha = \{t_1^{(\alpha)}, \ldots, t_{p_\alpha}^{(\alpha)}\}, \alpha = 1, \ldots, q, p_\alpha = |K_\alpha|$. Substituting (2.4) into (2.3) we arrive at the expression that has the following form :

$$\sum_{\substack{\text{over partitions} \\ \mathcal{M} = \{M_1, \ldots, M_r\} \text{ of } \{1, \ldots, N\}}} \sum_{\substack{\text{over partitions} \\ \mathcal{K} = \{K_1, \ldots, K_q\} \text{ of } \{1, \ldots, r\}}} \cdots .$$

To interchange the order of summation we construct a new partition $\mathcal{P} = \{P_1, \ldots, P_q\}$ of $\{1, 2, \ldots N\}$ as follows: $P_i = \sqcup_{j \in K_i} M_j, i = 1, \ldots, q.$ Then

$\{M_j\}_{j \in K_i}$ gives a partition of P_i that we denote by $\mathcal{P}_i$. We have

$$E_n\big(S_n(f)\big)^N = \sum_{\substack{\text{over partitions} \\ \mathcal{P}=\{P_1,\dots,P_q\} \text{ of } \{1,\dots,N\}}} \left(\prod_{i=1}^{q} \Big(\sum_{\substack{\text{over partitions} \\ \mathcal{P}_i \text{ of } P_i : \mathcal{P}_i = \{P_{i,1},\dots,P_{i,t_i}\}}} \right.$$

$$\int_{T^{t_i}} f^{|P_{i,1}|}(x_1) \cdot \ldots \cdot f^{|P_{i,t_i}|}(x_{t_i})(-1)^{t_i-1} \cdot \sum_{\substack{\text{over cyclic} \\ \text{permutations } \sigma \in S_{t_i}}} \tag{2.5}$$

$$\left. \prod_{j=1}^{t_i} Q_n(x_j, x_{\sigma(j)}) dx_1 \dots dx_{t_i} \Big) \right).$$

We remind that the moments are expressed in terms of cumulants as

$$m_N = \sum_{\substack{\text{over partitions} \\ \mathcal{P}=\{P_1,\dots,P_k\}}} C_{|P_1|} \cdot \ldots \cdot C_{|P_k|}.$$

Comparing the last formula with (2.5) we arrive at

$$C_{\ell,n}(f) = \sum_{\substack{\text{partitions} \\ \mathcal{P}=\{R_1,\dots,R_m\} \text{ of } \{1,\dots,\ell\}}} \int_{T^m} f^{|R_1|}(x_1) \cdot \ldots \cdot f^{|R_m|}(x_m) \cdot$$

$$(-1)^{m-1} \cdot \sum_{\substack{\text{cyclic permutations} \\ \sigma \in S_n}} \prod_{j=1}^{m} Q_n(x_j, x_{\sigma(j)})$$

$$= \sum_{m=1}^{\ell} \sum_{\substack{\text{over ordered collections} \\ (\ell_1,\dots,\ell_m): \sum_1^m \ell_i = \ell, \ell_i \geq 1}} (-1)^{m-1} \frac{\ell!}{\ell_1! \dots \ell_m!} \frac{1}{m!} \cdot \tag{2.6}$$

$$\int_{T^m} f^{\ell_1}(x_1) \cdot \ldots \cdot f^{\ell_m}(x_m) \cdot m! \cdot \frac{1}{m} \cdot$$

$$\prod_{j=1}^{m} Q_n(x_j, x_{j+1}) dx_1 \dots dx_m$$

$$= \sum_{m=1}^{\ell} \sum_{\substack{(\ell_1,\dots,\ell_m) : \ell_1 + \dots + \ell_m = \ell, \\ \ell_i \geq 1, i=1,\dots,m}}$$

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$$\frac{(-1)^{m-1}}{m} \cdot \frac{\ell!}{\ell_1! \cdot \ldots \cdot \ell_m!} \cdot \int_{T^m} f^{\ell_1}(x_1) \cdot \ldots \cdot f^{\ell_m}(x_m) \cdot$$
$$\prod_{j=1}^m Q_n(x_j, x_{j+1}) dx_1 \ldots dx_m. \tag{2.7}$$

Since $Q_n(x, y) = \sum_{j=0}^{n-1} e^{-ij(x-y)}$ we can rewrite (2.7) as

$$C_{\ell,n}(f) = \sum_{m=1}^{\ell} \sum_{\substack{(\ell_1, \ldots, \ell_m): \\ \ell_1 + \ldots + \ell_m = \ell, \ \ell_i \geq 1}} \frac{(-1)^{m-1}}{m} \cdot \frac{\ell!}{\ell_1! \ldots \ell_m!}$$

$$\sum_{s_1=0}^{n-1} \ldots \sum_{s_m=0}^{n-1} \widehat{f^{\ell_1}}(-s_m + s_1) \cdot \widehat{f^{\ell_2}}(-s_1 + s_2) \cdot \ldots \cdot \widehat{f^{\ell_m}}(-s_{m-1} + s_m).$$

Writing down the Fourier coefficients of the powers of f as the convolutions of the Fourier coefficients of f

$$\widehat{f^{\ell_1}}(-s_m + s_1) = \sum_{\substack{(k_1, \ldots, k_{\ell_1}): \\ k_1 + \ldots + k_{\ell_1} = s_1 - s_m}} \hat{f}(k_1) \cdot \ldots \cdot \hat{f}(k_{\ell_1}),$$

$$\widehat{f^{\ell_2}}(-s_1 + s_2) = \sum_{\substack{(k_{\ell_1+1}, \ldots, k_{\ell_2}): \\ k_{\ell_1+1} + \ldots + k_{\ell_2} = s_1 - s_2}} \hat{f}(k_{\ell_1+1}) \cdot \ldots \cdot \hat{f}(k_{\ell_2}), \cdots$$

$$\widehat{f^{\ell_m}}(-s_{m-1} + s_m) = \sum_{\substack{(k_{\ell_{m-1}+1}, \ldots, k_{\ell_m}): \\ k_{\ell_{m-1}+1} + \ldots + k_{\ell_m} = s_{m-1} - s_m}} \hat{f}(k_{\ell_{m-1}+1}) \cdot \ldots \cdot \hat{f}(k_{\ell_m}),$$

we obtain

$$C_{\ell,n}(f) = \sum_{k_1 + \ldots + k_\ell = 0} \hat{f}(k_1) \cdot \ldots \cdot \hat{f}(k_\ell) \cdot \sum_{m=1}^{\ell} \frac{(-1)^{m-1}}{m} \sum_{\substack{(\ell_1, \ldots, \ell_m): \\ \ell_1 + \ldots + \ell_m = \ell, \ \ell_i \geq 1}}$$

$$\frac{\ell!}{\ell_1! \ldots \ell_m!} \cdot \#\{u : 0 \leq u \leq n-1, 0 \leq u + \tag{2.8}$$

$$\sum_1^{\ell_1} k_i \leq n-1, \ldots, 0 \leq u + \sum_1^{\ell_1 + \ldots + \ell_{m-1}} k_i \leq n-1\}.$$

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The last factor in (2.8) is equal to

$$n - \max\left(0, \sum_1^{\ell_1} k_i, \ldots, \sum_1^{\ell_1 + \ldots + \ell_{m-1}} k_i\right) - \max\left(0, \sum_1^{\ell_1}(-k_i), \ldots, \sum_1^{\ell_1 + \ldots + \ell_{m-1}}(-k_i)\right)$$
(2.9)

if the expression in (2.9) is nonnegative or zero otherwise.

Lemma 1 is proven. □

3 Proof of the Main Combinatorial Lemma

First we show that $G(k_1, \ldots, k_\ell)$ is a linear combination of terms $|k_{i_1} + \ldots + k_{i_s}|$. Then we compute the coefficient in front of every such term and show it to be equal to zero.

Assume $\ell > 2$. Consider a partition $\mathcal{P} = \{P_1, \ldots, P_m\}$ of the set $\{1, 2, \ldots, \ell\}$. Let us denote $v_1 = \sum_{j \in P_1} k_j, \ldots, v_m = \sum_{j \in P_m} k_j$. The expression for G can be transformed into

$$G(k_1, \ldots, k_\ell) = \sum_{m=1}^{\ell} \sum_{\mathcal{P} = \{P_1, \ldots, P_m\}} \frac{(-1)^m}{m} \cdot \sum_{\tau \in S_m}$$
(3.1)

$$\max(0, v_{\tau(1)}, v_{\tau(1)} + v_{\tau(2)}, \ldots, v_{\tau(1)} + v_{\tau(2)} + \ldots + v_{\tau(m-1)}).$$

In [R-S] Rudnick and Sarnak, following the ideas of [K] and [Spi] (see also [B], [An]) , used the following identity for the set of real numbers $v_1, \ldots v_m$ with zero sum:

$$\frac{1}{m} \sum_{\tau \in S_m} \max(0, v_{\tau(1)}, v_{\tau(1)} + v_{\tau(2)}, \ldots, v_{\tau(1)} + v_{\tau(2)} + \ldots v_{\tau(m-1)})$$

$$= \frac{1}{4} \sum_{\substack{F \subset \{1, \ldots m\}, \\ F, F^C \neq \emptyset}} (|F| - 1)!(m - |F| - 1)! \cdot |\sum_{\ell \in F} v_\ell|$$
(3.2)

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The last formula gives us

$$G(k_1, \ldots, k_\ell) = \frac{1}{4} \sum_{m=1}^{\ell} \sum_{\mathcal{P}=\{P_1,\ldots,P_m\}} \sum_{\substack{F \subset \{1,\ldots m\}, \\ F, F^C \neq \emptyset}} (-1)^{|F|-1} \cdot$$

$$(|F|-1)! \cdot \left| \sum_{i \in \sqcup_{j \in F} P_j} k_i \right| \cdot (-1)^{(m-|F|-1)} \cdot (m-|F|-1)!. \tag{3.3}$$

Le us denote by A the subset $\sqcup_{j \in F} P_j$ of $\{1, 2, \ldots, \ell\}$. Then $\{P_j\}_{j \in F}$ defines a partition of A, and $\{P_j\}_{j \in F^C}$ a partition of $A^C = \{1, 2, \ldots, \ell\} \setminus A$.

We change now the order of summation in (3.3): first we sum over all nonempty subsets A of $\{1, 2, \ldots, \ell\}$ and then over all partitions of A and A^C:

$$G(k_1, \ldots, k_\ell) = \frac{1}{4} \sum_{\substack{A \subset \{1, \ldots, \ell\}, \\ A, A^C \neq \emptyset}} \cdot \left(\sum_{\substack{\text{over partitions} \\ \mathcal{U}=\{U_1,\ldots,U_r\} \text{ of } A}} (-1)^{|\mathcal{U}|-1}(|\mathcal{U}|-1)! \right) \cdot$$

$$\left(\sum_{\substack{\text{over partitions} \\ \mathcal{U}' \text{ of } A^C}} (-1)^{|\mathcal{U}'|-1} \cdot (|\mathcal{U}'|-1)! \right) \cdot \left| \sum_{i \in A} k_i \right|. \tag{3.4}$$

Finally we note that

$$\sum_{\mathcal{U}=\{U_1,\ldots U_r\}} (-1)^{|\mathcal{U}|-1} \cdot (|\mathcal{U}|-1)! = \sum_{r=1}^{|A|} \sum_{\substack{(t_1, \ldots, t_r): \\ \sum_{i=1}^{r} t_i=|A|, t_i \geq 1}} (-1)^{r-1} \cdot$$

$$(r-1)! \frac{|A|!}{t_1! \cdot \ldots \cdot t_r!} \cdot \frac{1}{r!},$$

the expression we already considered in (1.14). Indeed, there are exactly $\frac{|A|!}{t_1! \cdot \ldots \cdot t_r!} \cdot \frac{1}{r!}$ different partitions of A such that $\{t_1, \ldots, t_r\} = \{|U_1|, \ldots, |U_r|\}$. If $|A| \geq 2$ this sum is zero. If $|A| = 1$, then $|A^C| = \ell - |A| \geq 2$ and the second factor in (3.4) equals zero by the same argument. $\quad\square$

Now we turn to a combinatorial lemma first formulated in [Spo]. Let us denote by $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, $\beta = (\beta_1, \ldots, \beta_\ell)$ vectors with entries $\alpha_j \in \{0, 1\}$. We consider a lexicographic order on the set of such vectors: $\alpha < \beta$

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iff $\alpha_j \leq \beta_j$, $j = 1, \ldots, \ell$ and at least for one j_0 $\alpha_{j_0} < \beta_{j_0}$. Following [Spo] we call such nonzero vectors branches and a set T of ordered branches $T = \{\alpha^{(1)}, \ldots, \alpha^{(m)}\}$, $\alpha^{(1)} < \alpha^{(2)} < \ldots < \alpha^{(m)}$, $|T| = m < \ell$, a tree. We denote by $T(\ell)$ the set of all trees formed by a ℓ-dimensional vectors (branches). A combinatorial sum in question is

$$U(k_1, \ldots, k_\ell) = \sum_{T \in T(\ell)} (-1)^{|T|-1} \cdot \max(0, \alpha \cdot k | \alpha \in T) \qquad (3.5)$$

Here we used the notation $\alpha \cdot k = \sum_{j=1}^{\ell} \alpha_j \cdot k_j$. We call $\max(0, \alpha \cdot k | \alpha \in T)$ the maximum of the tree T. For a warm-up we prove

Proposition 1

$$U(k_1, \ldots, k_\ell) + U(-k_1, \ldots, -k_\ell) = G(k_1, \ldots, k_\ell, k_{\ell+1})$$
$$+ G(-k_1, \ldots, -k_\ell, -k_{\ell+1}),$$

where $k_{\ell+1} = -k_1 - k_2 - \ldots - k_\ell$.

Remark 7 Once the proposition is proven we see of course that $U(k_1, \ldots, k_\ell) + U(-k_1, \ldots, -k_\ell)$ is zero for $\ell \geq 2$.

Proof of Proposition 1 In the above notations

$$G(k_1, \ldots, k_\ell, k_{\ell+1}) = {\sum_{T \in T(\ell+1)}}' \frac{(-1)^{|T|-1}}{|T|} \cdot \max(0, \alpha \cdot k' | \alpha \in T),$$

where $k' = (k_1, \ldots, k_\ell, k_{\ell+1})$, and the sum $\sum'$ is over all trees $T \in T(\ell + 1)$ such that the largest branch of T, $\alpha^{(|T|)}$ is less than $D = (1, 1, \ldots, 1)$. Similarly, we can write $U(k_1, \ldots, k_\ell) = \sum''_{T \in T(\ell+1)} (-1)^{|T|-1} \cdot \max(0, \alpha \cdot k' | \alpha \in T)$, where the sum $\sum''$ is over the trees $T \in T(\ell + 1)$ such that the $(\ell + 1)^{th}$ coordinate of $\alpha^{(|T|)}$ is zero. We define a "rotation" on the set of all trees such that $\alpha^{(|T|)} \neq D$: $W((\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(|T|)})) = (\alpha^{(2)} - \alpha^{(1)}, \alpha^{(3)} - \alpha^{(1)}, \ldots, \alpha^{(|T|)} - \alpha^{(1)}, D - \alpha^{(1)})$. Since $\sum_{j=1}^{\ell+1} k_j = 0$, we observe that

$$\max(0, \alpha \cdot k' | \alpha \in T) + \max(0, \alpha \cdot (-k') | \alpha \in T) =$$
$$\max\left(0, \alpha \cdot k' | \alpha \in W(T)\right) + \max\left(0, \alpha \cdot (-k') | \alpha \in W(T)\right). \qquad (3.6)$$

The last equality implies

$$U(k_1,\ldots,k_\ell) + U(-k_1,\ldots,-k_\ell) = \sideset{}{''}\sum_{T\in T(\ell+1)} (-1)^{|T|-1} \cdot (\max(0,\alpha\cdot k'|\alpha\in T)$$
$$+ \max(0,\alpha\cdot(-k')|\alpha\in T))$$
$$= \sideset{}{''}\sum_{T\in T(\ell+1)} (-1)^{|T|-1}\cdot\frac{1}{|T|}\cdot\sum_{p=0}^{|T|-1}$$
$$(\max(0,\alpha\cdot k'|\alpha\in W^p(T))$$
$$+ \max(0,\alpha\cdot(-k')|\alpha\in W^p(T))$$
$$= \sum_{\substack{T\in T(\ell+1)\\ \alpha^{(|T|)}\neq D}} \frac{(-1)^{|T|-1}}{|T|}\cdot(\max(0,\alpha\cdot k'|\alpha\in T)$$
$$+ \alpha\cdot(-k')|\alpha\in T))$$
$$= G(k_1,\ldots,k_{\ell+1}) + G(-k_1,\ldots,-k_{\ell+1})$$

Here we used that for any T' with $\alpha^{(|T'|)}\neq D$ there exist a unique T with $\alpha_{\ell+1}^{(|T|)}=0$ and $0\leq p<|T|$ such that $T'=W^p(T)$. □

Proposition 2

$$U(k_1,\ldots,k_\ell) = 0 \text{ if } \ell\geq 2. \tag{3.7}$$

□

We proceed by induction.

It is easy to check the case $\ell=2$. Let us assume that the proposition is true for some $\ell\geq 2$. Consider $U(k_0,k_1,\ldots,k_\ell)$. Since U is a symmetric function we may assume $k_0\leq k_1\leq\ldots k_\ell$. The continuity of U implies that it is enough to check (3.7) for nondegenerate vectors $(k_0,k_1,\ldots,k_\ell)$. Therefore we may assume that the coordinates $k_1,\ldots k_\ell$ are linearly indepdendent over the integers. Fix such $k_1,\ldots k_\ell$ and consider U as a piecewise linear function of $y=k_0$, $U(y,k)=U(y,k_1,k_2,\ldots,k_\ell)$. Our first claim is that $U(y,k)$ is zero for all negative y. To show this we write

$$U(y,k) = \sum_{T\in T(\ell+1)} (-1)^{|T|-1}\max(0,\alpha\cdot(y,k)|\alpha\in T)$$
$$= \sum_{\substack{T\in T(\ell+1):\\ \alpha_1^{(|T|)}=0}} + \sum_{\substack{T\in T(\ell+1):\\ \alpha_1^{(|T|)}=1,\ \alpha_1^{(|T|)-1}=0}} + \sum_{\substack{T\in T(\ell+1):\\ \alpha_1^{(|T|)}=1,\ \alpha_1^{(|T|-1)}=1}}$$

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We denote the three subsums by U_1, U_2, U_3. The first subsum is equal to

$$\sum_{T \in T(\ell)} (-1)^{|T|-1} \cdot \max(0, \alpha \cdot k | \alpha \in T),$$

the second –

$$\sum_{T \in T(\ell)} (-1)^{|T|} \cdot \max(0, \alpha \cdot k | \alpha \in T) + \max(0, y),$$

and by the induction assumptions both are zero. Now we split the third subsum in two. Consider the smallest branch $\alpha \in T$ such that the first coordinate of α is 1, denote this branch by α' and denote the preceding (may be empty) branch by α''. We write $U_3 = U_{3,1} + U_{3,2}$, where in $U_{3,1}$ the summation is over $T \in T(\ell+1)$, such that $\alpha_1^{(|T|)} = 1$, $\alpha_1^{(|T|-1)} = 1$ and $\alpha' - \alpha'' > (1, 0, \ldots, 0)$, and in $U_{3,2}$ the summation is over all other trees from U_3. We establish a one-to-one correspondence between $U_{3,1}$ and $U_{3,2}$: for any tree T_1 with $\alpha' - \alpha'' > (1, 0, \ldots, 0)$ we construct $T_2 = \{\alpha^{(1)}, \ldots, \alpha'', \alpha'' + (1, 0, \ldots, 0), \alpha', \ldots, \alpha^{|T|}\}$. Clearly, $|T_2| = |T_1| + 1$, therefore

$$(-1)^{|T_1|-1} \cdot \max(0, \alpha \cdot (y, k) | \alpha \in T_1) = -(-1)^{|T_2|-1} \cdot \max(0, \alpha \cdot (y, k) | \alpha \in T_2),$$

and $U_{3,1}$ and $U_{3,2}$ cancel each other.

Now we assume that y is nonnegative and $0 \leq y \leq k_1 < k_2 < \ldots < k_\ell$. As we already noted $U(y, k_1, \ldots, k_\ell)$ is a piecewise linear continuous function. We claim that it can change its slope only at $y = 0$. Indeed, $U(y, k_1, \ldots, k_\ell)$ can change its slope only at the points of degeneracy of $(y, k_1, \ldots, k_\ell)$, where $\alpha_0 \cdot y + \alpha \cdot k = \alpha'_0 \cdot y + \alpha'_0 \cdot k$ and the coordinates of (α_0, α), (α'_0, α') take values zero and one. Because k is a non-degenerate vector we must have $y + \alpha \cdot k = \alpha' \cdot k$ (or $\alpha \cdot k = y + \alpha' k$). Since the tree T contains both branches $(1, \alpha)$ and $(0, \alpha')$ only if $\alpha' \leq \alpha$, the only solution for nonnegative vector (y, k) must be $y = 0$, $\alpha' = \alpha$. We will finish the proof of the proposition if we show that $U(y, k) = 0$ for sufficiently small positive y. We again write $U = U_1 + U_2 + U_3$ as before. Then $U_1 = 0$ by inductive assumption and U_3 is zero for sufficiently small y ($U_{3,1}$ and $U_{3,2}$ still cancel each other). We can write the second subsum U_2 as

$$\sum_{T \in T(\ell)} (-1)^{|T|} \cdot \left(\max(0, \alpha \cdot k | \alpha \in T) + y \right) =$$

$$\sum_{T \in T(\ell)} (-1)^{|T|} \cdot \left(\max(0, \alpha \cdot k | \alpha \in T) \right) + y \cdot \sum_{T \in T(\ell)} (-1)^{|T|}$$

(3.8)

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(the last sum includes empty tree). The first term in (3.8) is zero by inductive assumption and the second is also zero since

$$\sum_{T \in T(\ell)} (-1)^{|T|} = \sum_{\substack{\ell_1 + \ldots + \ell_m = \ell + 1, \\ \ell_i \geq 1}} \frac{(-1)^{m-1}}{m} \cdot \frac{(\ell + 1)!}{\ell_1! \cdot \ldots \cdot \ell_m!} = 0.$$

Proposition 2 is proven. $\square$

4   Orthogonal and symplectic groups.

We start with the orthogonal case. The eigenvalues of matrix $M \in \mathrm{SO}(2n)$ can be arranged in pairs

$$\exp(i\theta_1), \exp(-i\theta_1), \ldots, \exp(i\theta_n), \exp(-i\theta_n), 0 \leq \theta_1, \theta_2, \ldots, \theta_n < \pi.$$

Consider the normalized Haar measure on $\mathrm{SO}(2n)$. The probability distribution of the eigenvalues is defined by its density (see [We]):

$$P_{\mathrm{SO}2n}(\theta_1, \ldots, \theta_n) = 2 \cdot \left(\frac{1}{2\pi}\right) \cdot \prod_{1 \leq i < j < \leq n} (2 \cos \theta_i - 2 \cos \theta_j)^2 \qquad (4.1)$$

The k-point correlation functions are given by (see [So1])

$$\rho_{n,k}(\theta_1, \ldots, \theta_k) = \det \left(K_{2n-1}^+(\theta_i, \theta_j) \right)_{1 \leq i,j \leq n} \qquad (4.2)$$

where

$$K_{2n-1}^+(x, y) = K_{2n-1}(x, y) + K_{2n-1}(x, -y) =$$
$$\frac{1}{2\pi} \cdot \left(\frac{\sin\left(\frac{(2n-1)(x-y)}{2}\right)}{\sin\left(\frac{x-y}{2}\right)} + \frac{\sin\left(\frac{(2n-1)(x+y)}{2}\right)}{\sin\left(\frac{x+y}{2}\right)} \right). \qquad (4.3)$$

In [D-S] and [Jo2] Diaconis-Shahshahani and Johansson studied asymptotic properties of linear statistics $\sum_{j=1}^n f(\theta_j)$ where for simplicity we may assume that f is real even trigonometric polynomial, $f(\theta) = \sum_{k=1}^m a_k(\ell^{ik\theta} +$

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$\ell^{-ik\theta}), a_k = \hat{f}(k), k = 1, 2, \ldots, m$. As before we denote the linear statistics by $S_n(f)$. Then $S_n(f) = \text{Trace } (\sum_{k=1}^{m} a_k M^k)$. It was shown that

$$E_{2n} \exp \left(t \cdot \sum_{j=1}^{n} f(\theta_j) \right) =$$
$$\exp \left(t\frac{1}{2} \sum_{k=1}^{m} (1 + (-1)^k) \hat{f}(k) + \frac{t^2}{2} \sum_{k=1}^{m} k\hat{f}(k)^2 + \bar{0}(1) \right) \tag{4.4}$$

which implies the convergence in distribution of $\sum_{j=1}^{n} f(\theta_j)$ to the normal law

$$N \left(\frac{1}{2} \cdot \sum_{k=1}^{m} \left(1 + (-1)^k\right) \hat{f}(k), \sum_{k=1}^{m} k \cdot \hat{f}(k)^2 \right).$$

(Actually (4.4) holds under much weaker conditions — it is enough to assume $f \in C^{1+\alpha}([0, \pi]), \alpha > 0$).

Remark 8 Similarly to the unitary case (4.4) is equivalent to the large n asymptotics result for some determinants, this time Hankel determinants (see [Jo2], [Jo1]).

Our combinatorial approach allows to prove CLT for all $f \in C^1([0, \pi])$ as well as to study the local linear statistics $\sum_{j=1}^{n} g(L_n \cdot (\theta_j - \theta)), 0 < \theta < \pi$. In particular we establish

Theorem 2 *Let g be a Schwartz function, $L_n \to +\infty, \frac{L_n}{n} \to 0$ and $0 < \theta < \pi$. Then $E_{2n} \sum_{j=1}^{n} g(L_n \cdot (\theta_j - \theta)) = \frac{n}{L_n \cdot \pi} \cdot \int_{-\infty}^{\infty} g(x)dx + \bar{0}(1)$, and the centralized random variable $\sum_{j=1}^{n} g(L_n \cdot (\theta_j - \theta)) - E_{2n} \sum_{j=1}^{n} g(L_n \cdot (\theta_j - \theta))$ converges in distribution to the normal law $N(0, \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{g}(t)|^2 |t| dt)$.*

Theorem 2 also holds for $SO(2n + 1)$ and $Sp(n)$.

Let $M \in SO(2n + 1)$. Then one of the eigenvalues of M is 1 and the other $2n$ eigenvalues can be arranged in pairs as before. The density of the eigenvalues is equal to

$$P_{SO(2n+1)}(\theta_1, \ldots, \theta_n) = \left(\frac{2}{\pi} \right)^2 \cdot \prod_{1 \leq i < j \leq n} (2\cos\theta_i - 2\cos\theta_j)^2 \cdot \prod_{i=1}^{n} \sin^2 \left(\frac{\theta_i}{2} \right). \tag{4.5}$$

The formula for the k-point correlation function is

$$\rho_{n,k}(\theta_1, \ldots, \theta_k) = \det \left(K_{2n}^-(\theta_i, \theta_j) \right)_{i,j=1,\ldots,k} \tag{4.6}$$

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where

$$K_{2n}^-(x, y) = K_{2n}(x, y) - K_{2n}(x, -y) =$$

$$\frac{1}{2\pi} \left(\frac{\sin(n(x - y))}{\sin\left(\frac{x-y}{2}\right)} - \frac{\sin(n(x + y))}{\sin\left(\frac{x+y}{2}\right)} \right). \tag{4.7}$$

The analogue of (4.4) reads

$$E_{2n+1} \left(\exp\left(t \sum_{j=1}^n f(\theta_j) \right) \right) =$$

$$\exp\left(t\frac{1}{2} \sum_{k=1}^m \left(-1 + (-1)^k\right) \hat{f}(k) + \frac{t^2}{2} \sum_{k=1}^m k\hat{f}(k)^2 + \bar{0}(1) \right). \tag{4.8}$$

In the symplectic case $M \in \mathrm{Sp}(n)$ the $2n$ eigenvalues again can be arranged in pairs

$$\exp(i \cdot \theta_i), \exp(-i \cdot \theta_1), \dots, \exp(i \cdot \theta_n), \exp(-i \cdot \theta_n), 0 \le \theta_1, \theta_2, \dots, \theta_n < \pi,$$

their density is equal to

$$P_{\mathrm{Sp}(n)}(\theta_1, \dots, \theta_n) = \left(\frac{2}{\pi}\right)^n \cdot \prod_{1 \le i < j \le n} (2\cos\theta_i - 2\cos\theta_j)^2 \cdot \prod_{i=1}^n \sin^2(\theta_i), \tag{4.9}$$

and the formula for k-point correlation function is

$$\rho_{n,k}(\theta_1, \dots, \theta_k) = \det\left(K_{2n+1}^-(\theta_i, \theta_j)\right)_{i,j=1,\dots k.} \tag{4.10}$$

The analogue of (4.4) reads

$$E_n \left(\exp(t \sum_{j=1}^n f(\theta_j)) \right) =$$

$$\exp\left(-t\frac{1}{2} \sum_{k=1}^m \left(1 + (-1)^k\right) \hat{f}(k) + \frac{t^2}{2} \sum_{k=1}^m k\hat{f}(k)^2 + \bar{0}(1). \right) \tag{4.11}$$

We will prove Theorem 2 for $\mathrm{SO}(2n)$. The proofs for $\mathrm{SO}(2n + 1)$ and $\mathrm{Sp}(n)$ are almost identical.

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Proof of Theorem 2 The arguments from §1 imply that it is enough to prove

Lemma 3 *Let $C_{\ell,n}(f)$ be the ℓ-th cumulant of $\sum_{j=1}^n f(\theta_j), \ell \geq 2$. Then*

$$\left| C_{\ell,n}(f) - \sum_{k_1+\ldots+k_\ell=0} \hat{f}(k_i) \cdot \ldots \cdot \hat{f}(k_\ell) \cdot \frac{1}{2}\Big(G(k_1,\ldots,k_\ell) \right.$$

$$\left. + G(-k_1,\ldots,-k_\ell)\Big)\right| \leq \mathrm{const}_\ell \sum_{\substack{k_1+\ldots+k_\ell=0 \\ |k_1|+\ldots+|k_\ell|>n}} \tag{4.12}$$

$$|k_1||\hat{f}(k_1)| \cdot \ldots \cdot |\hat{f}(k_\ell)| + \mathrm{const}'_\ell \sum_{|k_1|+\ldots+|k_\ell|>n} |\hat{f}(k_1)| \cdot \ldots \cdot |\hat{f}(k_\ell)|$$

$$\square$$

We start with the formula (2.6) which holds for general determinantal random point fields:

$$C_{\ell,n}(f) = \sum_{m=1}^{\ell} \sum_{\substack{\ell_1+\ldots+\ell_m=\ell, \\ \ell_i \geq 1}} (-1)^m \frac{\ell!}{\ell_1! \cdot \ldots \cdot \ell_m!} \cdot \frac{1}{m} \int_{[0,\pi]^m} f^{\ell_1}(x_1) \cdot \ldots \cdot f^{\ell_m}(x_m) \cdot$$

$$\prod_{j=1}^{m} \left(K_{2n-1}(x_j, x_{j+1}) + K_{2n-1}(x_j, -x_{j+1})\right) dx_1 \ldots dx_m$$

(we always assume $x_{m+1} = x_1$).

$$= \sum_{m=1}^{\ell} \sum_{\substack{\ell_1+\ldots+\ell_m=\ell \\ \ell_i \geq 1}} (-1)^m \cdot \frac{1}{m} \cdot \frac{\ell!}{\ell_1! \cdot \ldots \cdot \ell_m!} \sum_{\epsilon_1=\pm 1} \sum_{\epsilon_2=\pm 1} \cdots \sum_{\epsilon_m=\pm 1}$$

$$\int_{[0,\pi]^m} f^{\ell_1}(x_1) \cdot \ldots \cdot f^{\ell_m}(x_m) \cdot \prod_{j=1}^{m} K_{2n-1}(x_j, \epsilon_j \cdot x_{j+1}) dx_1 \cdot \ldots \cdot dx_m \tag{4.13}$$

Each term in the last sum with $\prod_{i=1}^{m} \epsilon_i = 1$ is equal to

$$\int_{\prod_{i=1}^{m} \epsilon_{i-1} \cdot [0,\pi]} f^{\ell_1}(x_1) \cdot \ldots \cdot f^{\ell_m}(x_m) \cdot \prod_{j=1}^{m} K_{2n-1}(x_j, x_{j+1}) \prod_{i=1}^{m} d(\epsilon_{i-1} \cdot x_i) =$$

$$\frac{1}{2^m} \cdot \int_{[0,2\pi]^m} f^{\ell_1}(x_1) \cdot \ldots \cdot f^{\ell_m}(x_m) \cdot \prod_{j=1}^{m} K_{2n-1}(x_j, x_{j+1}) dx_1 \cdot \ldots \cdot dx_m$$

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(we use the fact that $f(x)$ is even). Combining these terms together we obtain the same expression as for $\frac{1}{2} \cdot C_{\ell,2n-1}(\sum_{j=1}^{2n-1} f(\theta_j))$ in the case of $U(2n-1)$, which gives vanishing contribution if $\ell > 2$. Finally we claim that the contribution from the terms with $\prod_{i=1}^{m} \epsilon_i = -1$ can be bounded from above by

$$\text{const}'_\ell \cdot \sum_{|k_1| + \ldots + |k_\ell| > n} |\hat{f}(k_1)| \cdot \ldots \cdot |\hat{f}(k_\ell)|.$$

Indeed, the integral

$$\int_{[0,\pi]^m} f^{\ell_1}(x_1) \cdot \ldots \cdot f^{\ell_m}(x_m) \cdot \prod_{j=1}^{m} \left(\frac{1}{2\pi} \sum_{s_j=-n}^{n} e^{is_j(x_j - \epsilon_j \cdot x_{j+1})} \right) dx_1 \ldots dx_m$$

can be rewritten as

$$\frac{1}{2^m} \cdot \sum_{s_1=-n}^{n} \ldots \sum_{s_m=-n}^{n} \widehat{f^{\ell_1}}(s_1 - \epsilon_m \cdot s_m) \cdot \widehat{f^{\ell_2}}(s_2 - \epsilon_1 \cdot s_1) \cdot \ldots \cdot \widehat{f^{\ell_m}}(s_m - \epsilon_{m-1} \cdot s_{m-1})$$

Consider the euclidian basis $\{e_j\}_{j=1}^{m}$ in $\mathbb{R}^m$ and define $f_j = e_j - \epsilon_{j-1} e_{j-1}$, $\epsilon_0 = \epsilon_m$. The vectors $\{f_j\}_{j=1}^{m}$ form a basis in $\mathbb{R}^m$ iff $\prod_{j=1}^{m} \epsilon_j = -1$. Then for any m-tuple $(t_1, \ldots, t_m)$ there exists the only m-tuple $(s_1, \ldots, s_m)$ such that $t_j = s_j - \epsilon_{j-1} \cdot s_{j-1}$, $j = 1, \ldots, m$. We write $\widehat{f^{\ell_j}}(t_j) = \sum \hat{f}(k_{\ell_1 + \ldots + \ell_{j-1} + 1}) \cdot \ldots \cdot \hat{f}(k_{\ell_1 + \ldots + \ell_j})$, where the sum is over k_i such that $\sum_{\ell_1 + \ldots + \ell_{j-1} + 1}^{\ell_1 + \ldots + \ell_j} k_i = t_j$. When we plug this into (4.13) we obtain a linear combination of

$$\hat{f}(k_1) \cdot \ldots \cdot \hat{f}(k_m) \tag{4.14}$$

It is easy to see that for $|k_1| + \ldots + |k_m| \le n$ the coefficient with the term (4.14) is equal to

$$\frac{1}{2^m} \cdot \sum_{m=1}^{\ell} \sum_{\substack{\ell_1 + \ldots + \ell_m = \ell, \\ \ell_i \ge 1}} (-1)^m \cdot \frac{1}{m} \cdot \frac{\ell!}{\ell_1! \cdot \ldots \cdot \ell_m!} = 0$$

For $|k_1| + \ldots + |k_m| > n$ the coefficient is bounded from above by some constant. This finished the proof of Lemma 3. $\qquad \square$

Similar to §1 we obtain the proof of Theorem 2 by applying the lemma to $\sum_{j=1}^{n} g(L_n \cdot (\theta_j - \theta))$.

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